

## THE BGG RESOLUTION, CHARACTER AND DENOMINATOR FORMULAS, AND RELATED RESULTS FOR KAC-MOODY ALGEBRAS<sup>1</sup>

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**ABSTRACT.** Let  $\mathfrak{g}$  be a Kac-Moody algebra defined by a (not necessarily symmetrizable) generalized Cartan matrix. We construct a BGG-type resolution of the irreducible module  $L(\lambda)$  with dominant integral highest weight  $\lambda$ , and we use this to obtain character and denominator formulas analogous to those of Weyl. We also determine a condition on the algebra which is sufficient for these formulas to take their classical form, and which implies that the set of defining relations is complete.

**1. Introduction.** In 1968, V. G. Kac [4] and R. V. Moody [8] introduced certain infinite-dimensional Lie algebras defined by generalized Cartan matrices. These Kac-Moody algebras are a straightforward generalization of the finite-dimensional split semisimple algebras over a field of characteristic zero, and are constructed using generators and relations analogous to those in the finite-dimensional case. It seems reasonable, therefore, to hope that many of the classical results for the finite-dimensional algebras also hold for the Kac-Moody algebras. These include Weyl's character formula and denominator formula.

In [1], I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand give a very beautiful and satisfying proof of these formulas in the classical case by constructing an exact resolution, now called the BGG resolution, of a finite-dimensional irreducible module by Verma modules. The formulas were proved for irreducible modules with dominant integral highest weight over certain Kac-Moody algebras (those defined by symmetrizable generalized Cartan matrices) in [5]. In [3], it is shown that the BGG resolution can also be obtained in this case, giving another proof of Weyl's character and denominator formulas.

In the present paper, we construct a BGG-type resolution for an irreducible module with dominant integral highest weight over an arbitrary Kac-Moody algebra. We then use this to obtain formulas analogous to those of Weyl. These form the content of Corollaries 5.8 and 5.9. We also determine a condition on the algebra which would allow these formulas to take their classical form. In addition, we show that this condition would also imply that the set of defining relations for the algebra is in some sense complete, i.e. the radical of the algebra is zero.

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The paper is organized as follows. In §2, the basic definitions and notations are introduced, as well as several known results which will be needed in the sequel. In §3, we prove the decomposition theorem which is the main tool in the construction of the BGG resolution, which is given in §4. From the BGG resolution, we obtain character and denominator formulas in §5. In §6, we show that the question of the classical character and denominator formulas, as well as the problem of the radical, can be reduced to a condition on Verma module imbeddings.

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**2. Definitions and known results.** Let  $A = (A_{ij})$  be an  $l \times l$  generalized Cartan matrix (GCM), i.e. a matrix with integer entries satisfying  $A_{ii} = 2$  for  $1 \leq i \leq l$ ,  $A_{ij} \leq 0$  for  $i \neq j$ , and  $A_{ij} = 0$  if and only if  $A_{ji} = 0$ . Let  $K$  be a field of characteristic zero, and let  $\mathfrak{g}$  be any Lie algebra over  $K$  satisfying:

(1)  $\mathfrak{g}$  is generated by a finite-dimensional abelian subalgebra  $\mathfrak{h}$ , called the Cartan subalgebra, and elements  $e_1, \dots, e_l, f_1, \dots, f_l$ , called simple root vectors and negative simple root vectors, respectively.

(2) There are linearly independent sets  $\{h_1, \dots, h_l\}$  in  $\mathfrak{h}$  and  $\{\alpha_1, \dots, \alpha_l\}$  in  $\mathfrak{h}^*$ , the dual vector space of  $\mathfrak{h}$ , such that  $A_{ij} = \alpha_j(h_i)$  for all  $1 \leq i, j \leq l$ . The  $\alpha_i$ 's are called the simple roots, and the  $h_i$ 's are called the simple dual roots.

(3)  $[e_i, f_j] = \delta_{ij} h_i$  for all  $1 \leq i, j \leq l$ .

(4)  $[h, e_i] = \alpha_i(h) e_i$  and  $[h, f_i] = -\alpha_i(h) f_i$  for all  $h \in \mathfrak{h}$  and all  $1 \leq i \leq l$ .

(5)  $\text{ad}(e_i)^{1-A_{ii}}(e_j) = 0 = \text{ad}(f_i)^{1-A_{ii}}(f_j)$  for all  $i \neq j$ .

(6) There is an involutive antiautomorphism  $\eta: \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $\eta(e_i) = f_i$  for all  $1 \leq i \leq l$  and  $\eta(h) = h$  for all  $h \in \mathfrak{h}$ .

For each GCM  $A$ , such an algebra  $\mathfrak{g}$  exists and is called a GCM, or Kac-Moody, Lie algebra defined by  $A$ . These objects were introduced in [4 and 8].

Let  $\mathfrak{n}$  (respectively,  $\mathfrak{n}^-$ ) denote the subalgebra of  $\mathfrak{g}$  generated by  $e_1, \dots, e_l$  (respectively,  $f_1, \dots, f_l$ ). Set  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$  (vector space direct sum).

For  $i = 1, \dots, l$ , let  $\mathfrak{u}_i = Kh_i + Ke_i + Kf_i$ . It is easy to see from the relations that  $\mathfrak{u}_i$  is a subalgebra isomorphic to  $\text{sl}(2, K)$ .

We will occasionally use the notation  $\delta(-, -)$  for the Kronecker symbol, so that for any  $\lambda, \mu \in \mathfrak{h}^*$  we have  $\delta(\lambda, \mu) = 1$  if  $\lambda = \mu$  and  $\delta(\lambda, \mu) = 0$  if  $\lambda \neq \mu$ .

For any  $\mathfrak{h}$ -module  $M$  and any  $\lambda \in \mathfrak{h}^*$ , let  $M_\lambda = \{m \in M \mid h \cdot m = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}$ ; if  $M_\lambda \neq 0$ , call  $\lambda$  a weight of  $M$ . In case  $M$  is an  $\mathfrak{h}$ -module satisfying  $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$  and all  $M_\lambda$  are finite-dimensional, call  $M$  a weight module. In this case, we write  $\Pi(M)$  for the set of weights of  $M$ .

Let  $M$  be a  $\mathfrak{g}$ -module. If  $v \in M_\lambda$  is a nonzero vector such that  $\mathfrak{n} \cdot v = 0$ , then we call  $v$  a maximal vector of weight  $\lambda$ . If in addition  $M = U(\mathfrak{g}) \cdot v$  (where  $U(-)$  denotes the universal enveloping algebra functor), then we call  $v$  a highest weight vector of weight  $\lambda$ , and we say that  $M$  is a highest weight module of weight  $\lambda$ . Every highest weight module is a weight module.

Assume  $\lambda \in \mathfrak{h}^*$ , and let  $K(\lambda)$  be the one-dimensional  $\mathfrak{b}$ -module whose underlying space is  $K$ , and whose  $\mathfrak{b}$ -module structure is defined by  $(h + x) \cdot k = \lambda(h)k$  for

all  $h \in \mathfrak{h}$ ,  $x \in \mathfrak{n}$ , and  $k \in K$ . The induced  $\mathfrak{g}$ -module  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K(\lambda)$  is called the Verma module of weight  $\lambda$ . It is well known that  $M(\lambda)$  is the universal highest weight module of weight  $\lambda$ , and that  $M(\lambda)$  has a unique irreducible quotient, which we denote by  $L(\lambda)$ . Furthermore, any nonzero  $\mathfrak{g}$ -module homomorphism from  $M(\mu)$  to  $M(\lambda)$ , where  $\mu$  and  $\lambda$  are in  $\mathfrak{h}^*$ , is injective.

Define the root lattice  $Q$  to be the free abelian group with basis  $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathfrak{h}^*$ , and let  $Q^+ = \{\sum_{i=1}^l k_i \alpha_i \mid k_i \in \mathbf{Z}_+ \text{ for all } i\}$ , where  $\mathbf{Z}_+$  denotes the set of nonnegative integers. Define a partial order  $\leq$  on  $\mathfrak{h}^*$  by  $\mu \leq \lambda$  if and only if  $\lambda - \mu \in Q^+$ . For  $\alpha = \sum_{i=1}^l k_i \alpha_i \in Q^+$ , we define the height of  $\alpha$  by  $\text{ht}(\alpha) = \sum_{i=1}^l k_i$ .

The roots of  $\mathfrak{g}$  are defined to be those  $\alpha \in \mathfrak{h}^* \setminus \{0\}$  such that  $g_\alpha \neq 0$ . Let  $\Delta$  be the set of roots of  $\mathfrak{g}$ . From the relations, it is easy to see that  $\Delta = \Delta^+ \cup \Delta^-$ , where  $\Delta^+ = \Delta \cap Q^+$  is the set of positive roots and  $\Delta^- = -\Delta^+$  is the set of negative roots.

We define the partition function  $\mathcal{P}: Q \rightarrow \mathbf{Z}_+$  by  $\mathcal{P}(\alpha) = \dim U(\mathfrak{n})_\alpha$ . Note that  $\mathcal{P}(\alpha) = 0$  unless  $\alpha \in Q^+$ .

Let  $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbf{Z} \text{ for all } i\}$ . We call  $P$  the set of integral weights, and the set  $P^+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbf{Z}_+ \text{ for all } i\}$  the set of dominant integral weights.

For simplicity of notation, in what follows we will write  $\text{Hom}(-, -)$  for  $\mathfrak{g}$ -module homomorphisms and  $\text{Ext}(-, -)$  for equivalence classes of extensions in the category of weight modules.

Occasionally, certain types of series of submodules will be useful. We define here three of these.

**DEFINITION 2.1.** Let  $M$  be a weight module. A highest weight series (HWS) for  $M$  is a filtration  $0 = M_0 \subset M_1 \subset \dots$  of submodules of  $M$  such that  $M = \bigcup_{i \geq 0} M_i$  and each  $M_{i+1}/M_i$  is a highest weight module.

**DEFINITION 2.2.** Let  $M$  be a weight module. A Verma series (VS) for  $M$  is a filtration  $0 = M_0 \subset M_1 \subset \dots$  of submodules of  $M$  such that  $M = \bigcup_{i \geq 0} M_i$  and each  $M_{i+1}/M_i \simeq M(\lambda_i)$  for some  $\lambda_i \in \mathfrak{h}^*$ . In this case, we write  $(M: M(\mu))$  for the number of  $i$  such that  $\lambda_i = \mu$ , which is finite since  $M$  is a weight module.

**REMARK.** When  $M$  has a VS, the numbers  $(M: M(\mu))$  are well defined, which may be seen by character considerations.

**DEFINITION 2.3 [2].** Let  $M$  be a weight module, all of whose weights are  $\leq \lambda$ , and let  $\mu \leq \lambda$ . By a local composition series for  $M$  at  $\mu$  we mean a filtration of submodules  $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$  such that each factor  $F_i = M_i/M_{i-1}$  satisfies either  $F_i \simeq L(\mu_i)$  for some  $\mu_i \geq \mu$  or  $F_i$  is a weight module (not necessarily irreducible), none of whose weights are  $\geq \mu$ .

**REMARK.** In [2, Proposition 4.2], it is shown that local composition series always exist, for such  $M$  and  $\mu$ .

In [2 and 11] certain numbers  $(M: L(\mu))$  are defined for  $M$  and  $\mu$  as in Definition 2.3, which generalize the idea of multiplicity in a composition series. (As shown in [2],  $(M: L(\mu))$  is the number of  $L(\mu)$  which occur as factors in a local composition series for  $M$  at  $\mu$ .) It is sufficient for our purposes to know that  $(M: L(\mu)) \neq 0$  if and only if  $L(\mu)$  is a subquotient of  $M$  [11, Remark following Definition 5.2].

We state here some known results on  $\text{Hom}(-, -)$ ,  $\text{Ext}(-, -)$ , and subquotients.

**PROPOSITION 2.4 [9, LEMMA 3.1].** *If  $M$  is a weight module with no weights  $> \lambda$ , then  $\text{Ext}(M(\lambda), M) = 0$ .*

PROPOSITION 2.5 [9, LEMMA 3.2]. *If  $\text{Ext}(M(\mu), L(\lambda)) \neq 0$ , then  $(M(\lambda) : L(\mu)) \neq 0$ .*

PROPOSITION 2.6 [9, THEOREM 4.1]. *For any  $\lambda, \mu \in \mathfrak{h}^*$ ,  $(M(\lambda) : L(\mu)) \neq 0$  if and only if  $\text{Hom}(M(\mu), M(\lambda)) \neq 0$ .*

We will be working with characters of  $\mathfrak{g}$ -modules occasionally. For this, we need a few facts about the ring of which the characters are elements. Let  $\mathcal{A}$  be the set of all formal sums  $\sum_{\mu \in \mathfrak{h}^*} a_\mu e^\mu$ , where  $a_\mu \in \mathbf{Z}$  and  $a_\mu = 0$  outside the union of a finite number of sets of the form  $\{\mu \in \mathfrak{h}^* | \mu \leq \lambda\}$ . Here, the  $e^\mu$  are to be thought of as formal exponentials, so that addition and multiplication in  $\mathcal{A}$  may be defined in the usual way. In this way,  $\mathcal{A}$  becomes a commutative, associative  $\mathbf{Z}$ -algebra with identity  $1 = e^0$ . Observe that for any  $\beta \in Q^+$ ,  $1 - e^{-\beta}$  is invertible, and its inverse is  $1 + e^{-\beta} + e^{-2\beta} + \dots$ .

If  $M$  is a weight module such that  $\Pi(M)$  is contained in the union of a finite number of sets of the form  $\{\mu \in \mathfrak{h}^* | \mu \leq \lambda\}$ , then we define the character of  $M$  to be  $\text{ch } M = \sum_{\mu \in \Pi(M)} \dim M_\mu e^\mu \in \mathcal{A}$ . For the case where  $M$  is a Verma module, we have  $\text{ch } M(\nu) = \sum_{\mu \leq \nu} \mathcal{P}(\nu - \mu) e^\mu$ .

We now turn to the Weyl group. First, we define the Weyl group and the dot action on  $\mathfrak{h}^*$ . We then give some known techniques for raising and lowering Verma module imbeddings to new Verma module imbeddings via the dot action on highest weights.

For  $i = 1, \dots, l$ , define the linear involution  $r_i: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  by  $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ . The Weyl group  $W$  is then defined to be the subgroup of  $\text{GL}(\mathfrak{h}^*)$  generated by  $r_1, \dots, r_l$ . If  $w \in W$ , define the length  $l(w)$  of  $w$  to be the smallest  $n$  such that  $w$  may be written  $w = r_{i_1} \cdots r_{i_n}$ .

Choose  $\rho \in \mathfrak{h}^*$  such that  $\rho(h_i) = 1$  for  $i = 1, \dots, l$ . In general,  $\rho$  is not unique, but we fix a particular choice.

For  $\lambda \in \mathfrak{h}^*$  and  $w \in W$ , define the dot action of  $w$  on  $\lambda$  by  $w \cdot \lambda = w(\lambda + \rho) - \rho$ . It is easy to see that this defines an action of  $W$  on  $\mathfrak{h}^*$ .

If  $w \in W$ , we set  $\Phi_w = \{\beta \in \Delta^+ | w^{-1}\beta \in \Delta^-\}$  and  $\langle \Phi_w \rangle = \sum_{\beta \in \Phi_w} \beta$ . It is well known that  $|\Phi_w| = l(w)$  and  $\langle \Phi_w \rangle = \rho - w\rho$  for all  $w \in W$  (cf., for example, [7, Propositions 2.8 and 2.12]).

LEMMA 2.7. *If  $\lambda(h_i) + 1 \in \mathbf{Z}_+$ , i.e. if  $r_i \cdot \lambda \leq \lambda$ , then*

$$\dim \text{Hom}(M(r_i \cdot \lambda), M(\lambda)) = 1.$$

PROOF. Let  $v$  be a highest weight vector for  $M(\lambda)$ . Since  $M(\lambda) = U(\mathfrak{n}^-)v$  and  $r_i \cdot \lambda = \lambda - (\lambda(h_i) + 1)\alpha_i$ , we see that  $\dim M(\lambda)_{r_i \cdot \lambda} = 1$  and  $M(\lambda)_{r_i \cdot \lambda}$  is spanned by  $f_i^{\lambda(h_i)+1}v$ . Thus  $\dim \text{Hom}(M(r_i \cdot \lambda), M(\lambda)) \leq 1$ . On the other hand, direct computation shows that  $f_i^{\lambda(h_i)+1}v$  is a maximal vector, so that  $\text{Hom}(M(r_i \cdot \lambda), M(\lambda))$  is nonzero, and the result follows.

DEFINITION 2.8. For  $i = 1, \dots, l$ , define maps  $\underline{r}_i: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  by

$$\underline{r}_i \cdot \lambda = \begin{cases} r_i \cdot \lambda, & \text{if } r_i \cdot \lambda \leq \lambda, \\ \lambda, & \text{otherwise.} \end{cases}$$

The  $r_i$  are to be thought of as lowering operators, reflecting down, when possible, by an integral multiple of  $\alpha_i$ .

**PROPOSITION 2.9.** *Let  $\lambda, \mu \in \mathfrak{h}^*$  and  $i \in \{1, \dots, l\}$ . If  $\text{Hom}(M(\mu), M(\lambda)) \neq 0$ , then  $\text{Hom}(M(r_i \cdot \mu), M(r_i \cdot \lambda)) \neq 0$ .*

**PROOF.** Fix an embedding  $M(\mu) \subseteq M(\lambda)$ . By Lemma 2.7, we may identify  $M(r_i \cdot \mu)$  and  $M(r_i \cdot \lambda)$  with submodules of  $M(\mu)$  and  $M(\lambda)$ , respectively. Let  $v$  be a highest weight vector for  $M(r_i \cdot \mu)$ . If  $v \notin M(r_i \cdot \lambda)$ , then  $(M(\lambda)/M(r_i \cdot \lambda) : L(r_i \cdot \mu)) \neq 0$ . But  $f_i$  acts locally nilpotently on  $M(\lambda)/M(r_i \cdot \lambda)$ , since it acts nilpotently on a highest weight vector, and hence  $f_i$  acts locally nilpotently on any subquotient of  $M(\lambda)/M(r_i \cdot \lambda)$ . Since, by the representation theory of  $\mathfrak{sl}(2, K)$ ,  $f_i$  does not act nilpotently on the highest weight vector of  $L(r_i \cdot \mu)$ , this gives a contraction, and therefore  $v \in M(r_i \cdot \lambda)$ , giving an imbedding  $M(r_i \cdot \mu) \subseteq M(r_i \cdot \lambda)$ .

The above proposition shows how Verma module imbeddings may be lowered to new Verma module imbeddings via the dot action on highest weights. For the analogous result on raising Verma module imbeddings, we need a definition.

**DEFINITION 2.10.** For  $i = 1, \dots, l$ , define maps  $\bar{r}_i: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  by

$$\bar{r}_i \cdot \lambda = \begin{cases} r_i \cdot \lambda, & \text{if } r_i \cdot \lambda \geq \lambda, \\ \lambda, & \text{otherwise.} \end{cases}$$

The  $\bar{r}_i$  are to be thought of as raising operators, reflecting up, when possible, by an integral multiple of  $\alpha_i$ . The following result is proved using Enright's completion functors, and follows by the same argument as that used in the proof of [11, Lemma 8.14].

**PROPOSITION 2.11.** *Let  $\lambda, \mu \in \mathfrak{h}^*$ , and  $i \in \{1, \dots, l\}$ . If  $\text{Hom}(M(\mu), M(\lambda)) \neq 0$ , then  $\text{Hom}(M(\bar{r}_i \cdot \mu), M(\bar{r}_i \cdot \lambda)) \neq 0$ . In any case,  $\dim \text{Hom}(M(\bar{r}_i \cdot \mu), M(\bar{r}_i \cdot \lambda)) \geq \dim \text{Hom}(M(\mu), M(\lambda))$ .*

**3. Decomposition of modules with HWS.** In this section we define, for any set of weights  $\Pi$ , an equivalence relation such that any module with a highest weight series whose highest weights are in  $\Pi$  decomposes into the direct sum of submodules corresponding to the equivalence classes in  $\Pi$ . This decomposition will be used in the next section to obtain the BGG resolution.

**LEMMA 3.1.** *Let  $\mu, \nu \in \mathfrak{h}^*$ . If  $M$  is a highest weight module of highest weight  $\nu$ , and  $\text{Ext}(M(\mu), M) \neq 0$ , then  $(M(\nu) : L(\mu)) \neq 0$ .*

**PROOF.** Let  $0 = M_0 \subset \dots \subset M_n = M$  be a local composition series for  $M$  at  $\mu$ , so that each factor  $F_i = M_i/M_{i-1}$  ( $i = 1, \dots, n$ ) either has no weights  $\geq \mu$  or is isomorphic to some  $L(\mu_i)$  with  $\mu_i \geq \mu$ . Note that in the latter case,  $(M : L(\mu_i)) \neq 0$ , and hence  $(M(\nu) : L(\mu_i)) \neq 0$ .

Now, since  $\text{Ext}(M(\mu), M) \neq 0$ , we must have  $\text{Ext}(M(\mu), F_i) \neq 0$  for some  $i$ . But then, by Proposition 2.4,  $F_i \cong L(\mu_i)$  with  $\mu_i > \mu$ . Applying Proposition 2.5, we have  $(M(\mu_i) : L(\mu)) \neq 0$ , and now Proposition 2.6 gives a chain of Verma module imbeddings  $M(\mu) \subset M(\mu_i) \subseteq M(\nu)$ . Thus  $(M(\nu) : L(\mu)) \neq 0$ .

**PROPOSITION 3.2.** *Let  $\mu, \nu \in \mathfrak{h}^*$ , and let  $M$  be a highest weight module of weight  $\mu$  and  $N$  be a highest weight module of weight  $\nu$ . If  $\text{Ext}(M, N) \neq 0$ , then  $M(\mu)$  and  $M(\nu)$  have a common irreducible subquotient.*

**PROOF.** Since  $M$  has highest weight  $\mu$ , and  $M(\mu)$  is the universal highest weight module of weight  $\mu$ , we have an epimorphism  $f: M(\mu) \rightarrow M$ . Letting  $J = \text{Ker } f$ , we obtain the short exact sequence  $0 \rightarrow J \rightarrow M(\mu) \rightarrow M \rightarrow 0$ . But then  $\text{Hom}(J, N) \rightarrow \text{Ext}(M, N) \rightarrow \text{Ext}(M(\mu), N)$  is exact, so that either  $\text{Hom}(J, N) \neq 0$  or  $\text{Ext}(M(\mu), N) \neq 0$ . If  $\text{Ext}(M(\mu), N) \neq 0$ , then, by Lemma 3.1, we have  $(M(\nu): L(\mu)) \neq 0$ , so that  $L(\mu)$  is a subquotient of both  $M(\mu)$  and  $M(\nu)$ . Otherwise, if  $\text{Hom}(J, N) \neq 0$ , then  $J$  and  $N$  have a common irreducible subquotient (any irreducible subquotient of the image of any nonzero map  $J \rightarrow N$ ) which is also a subquotient of both  $M(\mu)$  and  $M(\nu)$ .

**DEFINITION 3.3.** If  $\Pi$  is any subset of  $\mathfrak{h}^*$ , we define a reflexive, symmetric relation  $R_\Pi$  on  $\Pi$  by  $\mu R_\Pi \nu$  if  $M(\mu)$  and  $M(\nu)$  have a common irreducible subquotient  $L(\chi)$ . (We do not require  $\chi$  to be in  $\Pi$ .) We then define  $\sim_\Pi$  to be the transitive closure of  $R_\Pi$ , so that  $\mu \sim_\Pi \nu$  if there is some finite sequence  $\nu_1, \dots, \nu_n$  in  $\Pi$  such that  $\mu R_\Pi \nu_1 R_\Pi \dots R_\Pi \nu_n R_\Pi \nu$ . Let  $\mathcal{S}_\Pi$  be the set of equivalence classes of  $\Pi$  under the equivalence relation  $\sim_\Pi$ .

**THEOREM 3.4.** *Let  $\Pi \subseteq \mathfrak{h}^*$  and suppose  $M$  is a weight module with HWS  $0 = M_0 \subset M_1 \subset \dots$  with factors  $F_i = M_{i+1}/M_i$  of highest weight  $\lambda_i \in \Pi$  ( $i = 0, 1, \dots$ ). Then there is a  $\mathfrak{g}$ -module decomposition  $M = \bigoplus_{S \in \mathcal{S}_\Pi} M^S$ , where each  $M^S$  has a HWS whose factors are those  $F_i$  with  $\lambda_i \in S$ , in order.*

**PROOF.** We first show, by induction on  $k$ , that the conclusion holds for  $M_k$ , which has the HWS  $0 = M_0 \subset M_1 \subset \dots \subset M_k$ . This is clear for  $k = 0, 1$ . We suppose that it holds for  $k$ , where  $k > 0$ , and we construct a decomposition for  $M_{k+1}$ . Since  $M_k = \bigoplus_{S \in \mathcal{S}_\Pi} M_k^S$  by induction, we have the short exact sequence

$$(3.1) \quad 0 \rightarrow \bigoplus_{S \in \mathcal{S}_\Pi} M_k^S \rightarrow M_{k+1} \rightarrow F_k \rightarrow 0.$$

Let  $\lambda_k \in S_0 \in \mathcal{S}_\Pi$ . If (3.1) splits, then we are done by defining  $M_{k+1}^S = M_k^S$  for  $S \neq S_0$ , and  $M_{k+1}^{S_0} = M_k^{S_0} \oplus F_k$ . Thus, we may assume that (3.1) does not split, and hence  $\text{Ext}(F_k, \bigoplus_{S \in \mathcal{S}_\Pi} M_k^S) \neq 0$ . By additivity of  $\text{Ext}$ ,  $\text{Ext}(F_k, M_k^T) \neq 0$  for some  $T \in \mathcal{S}_\Pi$ . But now, from the HWS for  $M_k^T$ , there is some  $F_i$ ,  $i < k$ , with  $\lambda_i \in T$  and  $\text{Ext}(F_k, F_i) \neq 0$ . Applying Proposition 3.2, we see that  $M(\lambda_i)$  and  $M(\lambda_k)$  have a common irreducible subquotient, so that necessarily  $T = S_0$ , and  $\text{Ext}(F_k, \bigoplus_{S \neq S_0} M_k^S) = 0$ .

Let  $T = S_0$ , and set  $V = M_{k+1}/\bigoplus_{S \neq T} M_k^S$ . Identify  $M_k/\bigoplus_{S \neq T} M_k^S$  with  $M_k^T$  by the natural map. Then

$$\frac{V}{M_k^T} \cong \frac{M_{k+1}/\bigoplus_{S \neq T} M_k^S}{M_k/\bigoplus_{S \neq T} M_k^S} \cong \frac{M_{k+1}}{M_k} = F_k,$$

and we have the commutative diagram with exact rows

$$(3.2) \quad \begin{array}{ccccccccc} 0 & \rightarrow & M_k & \rightarrow & M_{k+1} & \rightarrow & F_k & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \rightarrow & M_k^T & \rightarrow & V & \rightarrow & F_k & \rightarrow & 0. \end{array}$$

On the other hand, the usual pushout construction gives the commutative diagram with exact rows

$$(3.3) \quad \begin{array}{ccccccccc} 0 & \rightarrow & M_k & \rightarrow & M_{k+1} & \rightarrow & F_k & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \rightarrow & \bigoplus_{S \neq T} M_k^S & \rightarrow & L & \rightarrow & F_k & \rightarrow & 0. \end{array}$$

Since  $\text{Ext}(F_k, \bigoplus_{S \neq T} M_k^S) = 0$ , we must have  $L \simeq F_k \oplus (\bigoplus_{S \neq T} M_k^S)$ , and projecting the bottom row of (3.3) onto the second summand gives the commutative diagram with exact rows

$$(3.4) \quad \begin{array}{ccccccccc} 0 & \rightarrow & M_k & \rightarrow & M_{k+1} & \rightarrow & F_k & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \bigoplus_{S \neq T} M_k^S & \rightarrow & \bigoplus_{S \neq T} M_k^S & \rightarrow & 0 & \rightarrow & 0. \end{array}$$

Adding the vertical maps in (3.2) and (3.4) gives finally the commutative diagram with exact rows

$$(3.5) \quad \begin{array}{ccccccccc} 0 & \rightarrow & M_k & \rightarrow & M_{k+1} & \rightarrow & F_k & \rightarrow & 0 \\ & & \downarrow & & \downarrow \phi & & \parallel & & \\ 0 & \rightarrow & \bigoplus_{S \in \mathcal{S}_{\Pi}} M_k^S & \rightarrow & V \oplus \left( \bigoplus_{S \neq T} M_k^S \right) & \rightarrow & F_k & \rightarrow & 0. \end{array}$$

By the Short Five Lemma,  $\phi$  is an isomorphism. Note that  $V$  has a HWS whose factors are those of  $M_k^T$ , together with  $F_k$  as the top factor. Thus, setting  $M_{k+1}^S = M_k^S$  for  $S \neq T$  and  $M_{k+1}^T = \phi^{-1}(V)$ , the conclusion holds for  $k+1$ , and the order of the factors is preserved.

Now, by the above construction, for all  $S \in \mathcal{S}_{\Pi}$  we have  $M_0^S \subseteq M_1^S \subseteq \cdots$ , so that we may set  $M^S = \bigcup_{k \geq 0} M_k^S$ . Then  $M = \bigoplus_{S \in \mathcal{S}_{\Pi}} M^S$ , since direct sums commute with direct limits.

**REMARK.** The preceding theorem is similar in spirit to a decomposition theorem of Deodhar, Gabber, and Kac [2, Theorem 4.2]. In fact, with  $\Pi = \mathfrak{h}^*$ , the statements are equivalent. Their result is not sufficient for our purposes, however, as follows. We will need to work with a very special set of weights  $\Pi$  in the next section. It is possible to have  $M(\chi) \subseteq M(\mu)$  and  $M(\chi) \subseteq M(\nu)$  with  $\mu, \nu \in \Pi$  but  $\chi \notin \Pi$ . In this situation, in agreement with [2], we require  $\mu$  and  $\nu$  to be in the same equivalence class, since otherwise the decomposition theorem would not be valid. On the other hand, it is also possible to have  $M(\mu) \subseteq M(\chi)$  and  $M(\nu) \subseteq M(\chi)$  with  $\mu, \nu \in \Pi$  and  $\chi \notin \Pi$ , while  $\mu \not\sim_{\Pi} \nu$ . In this case, in our decomposition we must have the highest weight factors of weight  $\mu$  occurring in a different summand from those of weight  $\nu$ , whereas they would occur in the same summand in the decomposition of [2].

**PROPOSITION 3.5.** *Let  $M$  and  $N$  satisfy the hypothesis of Theorem 3.4, and let  $\phi \in \text{Hom}(M, N)$ . Then  $\phi(M^S) \subseteq N^S$  for all  $S \in \mathcal{S}_{\Pi}$ .*

**PROOF.** Let  $\pi_T: N = \bigoplus_{S \in \mathcal{S}_{\Pi}} N^S \rightarrow N^T$  be the natural projection, where  $T \in \mathcal{S}_{\Pi}$ . We must show that, if  $S \neq T$ , then  $\pi_T \phi(M^S) = 0$ . If not, then  $\text{Hom}(M^S, N^T) \neq 0$ , so that  $\text{Hom}(F_i, G_j) \neq 0$  for some factor  $F_i$  of highest weight  $\lambda_i \in S$  in the HWS for  $M$ , and some factor  $G_j$  of highest weight  $\mu_j \in T$  in the HWS for  $N$ . But then  $M(\lambda_i)$  and  $M(\mu_j)$  have a common irreducible subquotient, contrary to the assumption that  $S \neq T$ . Thus  $\pi_T \phi(M^S) = 0$  for all  $S \neq T$ , and we have  $\phi(M^S) \subseteq N^S$  for all  $S \in \mathcal{S}_{\Pi}$ .

**COROLLARY 3.6.** *The decomposition  $M = \bigoplus_{S \in \mathcal{S}_{\Pi}} M^S$  is independent of the choice of HWS for  $M$  satisfying the hypothesis of Theorem 3.4.*

**PROOF.** Take  $\phi$  to be the identity map on  $M$  in Proposition 3.5.

**PROPOSITION 3.7.** *Let  $\Pi \subseteq \mathfrak{h}^*$ . For any  $S \in \mathcal{S}_{\Pi}$ , the functor  $M \mapsto M^S$  takes exact sequences of  $\mathfrak{g}$ -modules satisfying the hypothesis of Theorem 3.4 to exact sequences of  $\mathfrak{g}$ -modules.*

**PROOF.** If  $M_1, M_2$ , and  $M_3$  satisfy the hypotheses of the theorem and we have the exact sequence  $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ , then, by Proposition 3.5,  $\phi(M_1^S) \subseteq M_2^S$  and  $\psi(M_2^S) \subseteq M_3^S$ . Thus, by restriction, we have

$$(3.6) \quad M_1^S \xrightarrow{\phi^S} M_2^S \xrightarrow{\psi^S} M_3^S.$$

Now  $\psi^S \phi^S(M_1^S) = \psi \phi(M_1^S) = 0$ , so  $\text{Im } \phi^S \subseteq \text{Ker } \psi^S$ . Suppose  $m_2 \in \text{Ker } \psi^S$ . Then  $m_2 \in M_2^S \cap \text{Ker } \psi$ , so that  $m_2 \in \text{Im } \phi$ , and  $m_2 = \phi(m_1)$  for some  $m_1 \in M_1$ . Write  $m_1 = \sum_{T \in \mathcal{S}_{\Pi}} m_1^T$  according to the decomposition of  $M_1$ , so that  $m_1^T \in M_1^T$  for all  $T \in \mathcal{S}_{\Pi}$ . By Proposition 3.5,  $\phi(m_1^T) \in M_2^T$  for all  $T \in \mathcal{S}_{\Pi}$ , so that, by directness of the decomposition of  $M_2$ ,  $\phi(m_1^S) = m_2$  and  $\phi(m_1^T) = 0$  for  $T \neq S$ . Thus  $\text{Ker } \psi^S \subseteq \text{Im } \phi^S$ , and the sequence (3.6) is exact.

**REMARK.** For our choices of  $\Pi$  and  $S$  in the next section, the functor  $M \mapsto M^S$  will coincide, in the case of a symmetrizable GCM, with taking the generalized eigenspace of the Casimir operator corresponding to the eigenvalue on an irreducible highest weight module of dominant integral highest weight.

**4. The BGG resolution.** We are now in a position to construct a resolution of the irreducible module  $L(\lambda)$ , for  $\lambda \in P^+$ . The construction is based on that of [1]. We do not, however, have central characters or Casimir operators working for us. Instead, we apply the decomposition functor  $M \mapsto M^S$  introduced in the previous section, with a suitable choice of  $\Pi$  and  $S$ .

For  $k \in \mathbf{Z}_+$ , let  $D_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Lambda^k(\mathfrak{g}/\mathfrak{b})$ , and define maps  $d_k: D_k \rightarrow D_{k-1}$  for  $k \geq 1$  by

$$\begin{aligned} d_k(X \otimes \bar{Y}_1 \wedge \cdots \wedge \bar{Y}_k) &= \sum_{i=1}^k (-1)^{i+1} (XY_i \otimes \bar{Y}_1 \wedge \cdots \wedge \hat{\bar{Y}}_i \cdots \wedge \bar{Y}_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} (X \otimes [\bar{Y}_i, \bar{Y}_j] \wedge \bar{Y}_1 \wedge \cdots \wedge \hat{\bar{Y}}_i \cdots \wedge \hat{\bar{Y}}_j \cdots \wedge \bar{Y}_k) \end{aligned}$$

for any  $X \in U(\mathfrak{g})$  and any  $Y_1, \dots, Y_k \in \mathfrak{g}$ , where  $\bar{Y} = Y + \mathfrak{b} \in \mathfrak{g}/\mathfrak{b}$  for any  $Y \in \mathfrak{g}$ . Also define  $\varepsilon: D_0 \rightarrow K$  by letting  $\varepsilon(X)$  be the constant term of  $X$ , for all  $X \in U(\mathfrak{g})$ .

PROPOSITION 4.1 [1, THEOREM II.9.1]. *The sequence*

$$\cdots \rightarrow D_k \xrightarrow{d_k} D_{k-1} \rightarrow \cdots \rightarrow D_0 \xrightarrow{\varepsilon} K \rightarrow 0$$

is an exact sequence of  $\mathfrak{g}$ -modules, denoted  $V(\mathfrak{g}, \mathfrak{b})$ .

Let  $\lambda \in P^+$ . Since  $L(\lambda)$  is a vector space over  $K$ ,  $-\otimes_K L(\lambda)$  is an exact functor. Thus  $V(\mathfrak{g}, \mathfrak{b}) \otimes_K L(\lambda)$  is exact. But by [3, Proposition 1.7],

$$D_k \otimes_K L(\lambda) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Lambda^k(\mathfrak{g}/\mathfrak{b})) \otimes_K L(\lambda)$$

is naturally isomorphic to  $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\Lambda^k(\mathfrak{g}/\mathfrak{b}) \otimes_K L(\lambda))$ , and certainly  $K \otimes_K L(\lambda) \simeq L(\lambda)$ , so that, setting  $D_k^\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\Lambda^k(\mathfrak{g}/\mathfrak{b}) \otimes_K L(\lambda))$ , we have the following

PROPOSITION 4.2. *For  $\lambda \in P^+$ ,  $L(\lambda)$  has an exact resolution*

$$\cdots D_k^\lambda \rightarrow D_{k-1}^\lambda \rightarrow \cdots \rightarrow D_0^\lambda \rightarrow L(\lambda) \rightarrow 0.$$

PROPOSITION 4.3 [1, LEMMA II.9.5]. *Each  $D_k^\lambda$  has a VS (possibly infinite), and for any  $\mu \in \mathfrak{h}^*$ ,*

$$(D_k^\lambda: M(\mu)) = \dim(\Lambda^k(\mathfrak{g}/\mathfrak{b}) \otimes_K L(\lambda))_\mu = \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu.$$

We are now ready to apply the functors introduced in §3. Fix  $\lambda \in P^+$  and let  $\Pi = \Pi(\Lambda \mathfrak{n}^-) + \Pi(L(\lambda))$ . Let  $S \in \mathcal{S}_\Pi$  be the equivalence class under  $\sim_\Pi$  containing  $\lambda$ . Set  $C_k = (D_k^\lambda)^S$  for  $k \in \mathbf{Z}_+$ .

THEOREM 4.4. *For  $\lambda \in P^+$ ,  $L(\lambda)$  has an exact resolution*

$$\cdots \rightarrow C_k \rightarrow C_{k-1} \rightarrow \cdots \rightarrow C_0 \rightarrow L(\lambda) \rightarrow 0.$$

Each  $C_k$  has a VS with factors  $M(\mu)$ , where  $\mu \in S$ , and

$$(C_k: M(\mu)) = \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu.$$

PROOF. Exactness follows from Proposition 3.7. Note that  $0 \subset L(\lambda)$  is a HWS for  $L(\lambda)$ , so that  $L(\lambda)^S = L(\lambda)$ . Also, since a VS is a special type of HWS, and each  $D_k^\lambda$  has a VS with multiplicities given in Proposition 4.3, the result follows from Theorem 3.4.

It now remains to determine more precisely which weights lie in  $S$ , and, for  $\mu \in S$ , to determine  $\dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu$ . We first recall a few facts about the roots of  $\mathfrak{g}$ . Since  $\mathfrak{g}$  is a locally finite  $\mathfrak{a}_i$ -module for  $i = 1, \dots, l$ ,  $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$  for any  $\alpha \in \Delta$  and any  $w \in W$ . Also, if  $1 \leq i \leq l$ , then  $r_i(\alpha_i) = -\alpha_i$  and  $r_i$  permutes the elements of  $\Delta^+ \setminus \{\alpha_i\}$  and permutes the elements of  $\Delta^- \setminus \{-\alpha_i\}$ .

The next proposition and corollary are a refinement of [3, Proposition 2.8].

PROPOSITION 4.5. *If  $\mu \in \mathfrak{h}^*$  and  $i \in \{1, \dots, l\}$ , then*

$$\sum_{k \geq 0} \dim(\Lambda^{2k} \mathfrak{n}^-)_\mu = \sum_{k \geq 0} \dim(\Lambda^{2k+1} \mathfrak{n}^-)_{r_i \cdot \mu}.$$

PROOF. For  $n = 0, 1, \dots$  and  $\nu \in \mathfrak{h}^*$ , let  $S_{n,\nu} = \{f: \Delta^- \rightarrow \mathbf{Z}_+ \mid f(\alpha) \leq \dim \mathfrak{g}_\alpha \text{ for all } \alpha \in \Delta^-, f(\alpha) = 0 \text{ for all but finitely many } \alpha, \sum_{\alpha \in \Delta^-} f(\alpha) = n, \text{ and } \sum_{\alpha \in \Delta^-} f(\alpha)\alpha = \nu\}$ . Observe that  $\dim(\wedge^n \mathfrak{n}^-)_\nu = |S_{n,\nu}|$ , and that the  $S_{n,\nu}$ 's are pairwise disjoint. We will construct a map  $\phi: \bigcup_{n \geq 0; \nu \in \mathfrak{h}^*} S_{n,\nu} \rightarrow \bigcup_{n \geq 0; \nu \in \mathfrak{h}^*} S_{n,\nu}$  which restricts to a bijection  $\bigcup_{k \geq 0} S_{2k,\mu} \rightarrow \bigcup_{k \geq 0} S_{2k+1,r_i \cdot \mu}$ , which will prove the assertion.

Let  $f \in S_{n,\nu}$ . Note that since  $\dim \mathfrak{g}_{-\alpha_i} = 1$ , we have  $f(-\alpha_i) = 0$  or 1. Suppose  $f(-\alpha_i) = 0$ . Then

$$\begin{aligned} r_i \cdot \nu &= r_i \left( \rho + \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)\alpha \right) - \rho \\ &= \rho - \alpha_i + \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)r_i(\alpha) - \rho \\ &= -\alpha_i + \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)r_i(\alpha) \end{aligned}$$

and, for  $\alpha \in \Delta^- \setminus \{-\alpha_i\}$ , we also have  $r_i(\alpha) \in \Delta^- \setminus \{-\alpha_i\}$ . Thus, we define  $\phi f \in S_{n+1,r_i \cdot \nu}$  by  $\phi f(-\alpha_i) = 1$  and  $\phi f(r_i(\alpha)) = f(\alpha)$  for  $\alpha \in \Delta^- \setminus \{-\alpha_i\}$ . On the other hand, if  $f(-\alpha_i) = 1$ , then

$$\begin{aligned} r_i \cdot \nu &= r_i \left( \rho - \alpha_i + \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)\alpha \right) - \rho \\ &= \rho - \alpha_i + \alpha_i + \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)r_i(\alpha) - \rho \\ &= \sum_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} f(\alpha)r_i(\alpha), \end{aligned}$$

and we define  $\phi f \in S_{n-1,r_i \cdot \nu}$  by  $\phi f(-\alpha_i) = 0$  and  $\phi f(r_i(\alpha)) = f(\alpha)$  for  $\alpha \in \Delta^- \setminus \{-\alpha_i\}$ . Now  $\phi$  restricts to maps  $\bigcup_{k \geq 0} S_{2k,\mu} \rightarrow \bigcup_{k \geq 0} S_{2k+1,r_i \cdot \mu}$  and  $\bigcup_{k \geq 0} S_{2k+1,r_i \cdot \mu} \rightarrow \bigcup_{k \geq 0} S_{2k,\mu}$ . But clearly  $\phi$  is an involution, so that these restrictions are bijective.

COROLLARY 4.6. *The set  $\rho + \Pi$  is  $W$ -invariant.*

PROOF. It suffices to show that if  $1 \leq i \leq l$ ,  $\mu \in \Pi(\wedge \mathfrak{n}^-)$ , and  $\nu \in \Pi(L(\lambda))$ , then  $r_i(\rho + \mu + \nu) \in \rho + \Pi$ . Let  $v$  be a highest weight vector for  $L(\lambda)$ . Since  $\lambda \in P^+$ ,  $f_i$  acts nilpotently on  $v$ , so that  $f_i$  acts locally nilpotently on  $L(\lambda)$ , since  $v$  generates  $L(\lambda)$ . Thus  $L(\lambda)$  is a locally finite  $\mathfrak{a}_i$ -module, and  $\dim L(\lambda)_\nu = \dim L(\lambda)_{r_i(\nu)}$ . In particular,  $r_i(\nu) \in \Pi(L(\lambda))$ . But, by the above proposition, we also have  $r_i \cdot \mu \in \Pi(\wedge \mathfrak{n}^-)$ , so that  $r_i(\rho + \mu + \nu) = \rho + r_i \cdot \mu + r_i(\nu) \in \rho + \Pi$ .

REMARK. The first argument in the above proof actually extends to show that  $\dim L(\lambda)_\nu = \dim L(\lambda)_{w(\nu)}$  for all  $\nu \in \Pi(L(\lambda))$  and all  $w \in W$ .

COROLLARY 4.7. *For any  $\mu \in \Pi$  and any  $w \in W$ ,*

$$\sum_{k \geq 0} (-1)^k \dim(\wedge^k \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \mu} = (-1)^{l(w)} \sum_{k \geq 0} (-1)^k \dim(\wedge^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu.$$

PROOF. It suffices, by induction on  $l(w)$ , to prove this when  $w = r_i$  for some  $i = 1, \dots, l$ . We have

$$\begin{aligned} \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_{r_i \cdot \mu} \\ &= \sum_{\chi \leq \lambda} \dim L(\lambda)_\chi \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^-)_{r_i \cdot \mu - \chi} \\ &= \sum_{\chi \leq \lambda} \dim L(\lambda)_{r_i(\chi)} \sum_{k \geq 0} (-1)^{k+1} \dim(\Lambda^k \mathfrak{n}^-)_{r_i \cdot (r_i \cdot \mu - \chi)} \end{aligned}$$

by the preceding remark and Proposition 4.5. But

$$\begin{aligned} r_i \cdot (r_i \cdot \mu - \chi) &= r_i \cdot (r_i(\mu + \rho) - \rho - \chi) \\ &= r_i(r_i(\mu + \rho) - \chi) - \rho = \mu + \rho - r_i(\chi) - \rho = \mu - r_i(\chi). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_{r_i \cdot \mu} \\ &= \sum_{\chi \leq \lambda} \dim L(\lambda)_{r_i(\chi)} \sum_{k \geq 0} (-1)^{k+1} \dim(\Lambda^k \mathfrak{n}^-)_{\mu - r_i(\chi)} \\ &= - \sum_{\chi \leq \lambda} \dim L(\lambda)_\chi \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^-)_{\mu - \chi} \\ &= - \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu. \end{aligned}$$

LEMMA 4.8. *If  $\mu \in \Pi$ , then, for some  $w \in W$ ,  $w(\mu + \rho) \in P^+ \cap (\rho + \Pi)$ .*

PROOF. By Corollary 4.6,  $w \cdot \mu \in \Pi$  for all  $w \in W$ . But clearly  $w \cdot \mu \leq \lambda$  for all  $w \in W$ , by definition of  $\Pi$ . Thus, we may choose  $w \in W$  such that  $\text{ht}(\lambda - w \cdot \mu)$  is minimal. Then  $w(\mu + \rho) = \rho + w \cdot \mu \in \rho + \Pi$ , and we must have  $w(\mu + \rho)(h_i) \geq 0$  for all  $i = 1, \dots, l$ , since otherwise  $(r_i w) \cdot \mu > w \cdot \mu$  would contradict the minimality of  $\text{ht}(\lambda - w \cdot \mu)$ . Thus  $w(\mu + \rho) \in P^+ \cap (\rho + \Pi)$ .

Although the following result is well known, the usual proof makes use of an invariant bilinear form, which does not exist in the nonsymmetrizable case. The following proof is a modification of the proof of [11, Lemma 8.2].

LEMMA 4.9. *Let  $\lambda \in P^+$ ,  $w \in W$ , and  $i \in \{1, \dots, l\}$ . Then  $l(r_i w) > l(w)$  if and only if  $r_i w \cdot \lambda < w \cdot \lambda$ . Also, if  $\lambda + \rho \in P^+$ ,  $w \in W$ , and  $i \in \{1, \dots, l\}$ , then  $l(r_i w) > l(w)$  implies  $r_i w \cdot \lambda \leq w \cdot \lambda$ .*

PROOF. We first assume that  $\lambda \in P^+$  and prove the first statement. Suppose  $r_i w \cdot \lambda < w \cdot \lambda$ , so that  $w \cdot \lambda - r_i w \cdot \lambda = n\alpha_i$  for some positive integer  $n$ . Then  $\lambda - w^{-1}r_i w \cdot \lambda = nw^{-1}\alpha_i$ . But

$$\lambda - w^{-1}r_i w \cdot \lambda = \lambda - w^{-1}r_i w(\lambda) + \langle \Phi_{w^{-1}r_i w} \rangle > 0,$$

since  $w^{-1}r_i w(\lambda) \in \Pi(L(\lambda))$ . Thus  $w^{-1}\alpha_i \in \Delta^+$ , and we have  $\alpha_i \notin \Phi_w$ ,  $\alpha_i \in \Phi_{r_i w}$ . Also for any  $\beta \in \Phi_w$ , we have  $r_i \beta \in \Delta^+ \setminus \{\alpha_i\}$  and  $(r_i w)^{-1}r_i \beta \in \Delta^-$ , so that  $\Phi_{r_i w} \supseteq \{\alpha_i\} \cup \{r_i \beta \mid \beta \in \Phi_w\}$ . Thus  $l(r_i w) = |\Phi_{r_i w}| > |\Phi_w| = l(w)$ .

Conversely, suppose  $l(r_i w) > l(w)$ . Observe that  $r_i w \cdot \lambda \neq w \cdot \lambda$ , since otherwise we would have

$$0 = w \cdot \lambda - r_i w \cdot \lambda = \lambda - w^{-1} r_i w \cdot \lambda = \lambda - w^{-1} r_i w(\lambda) + \langle \Phi_{w^{-1} r_i w} \rangle,$$

so that  $w^{-1} r_i w = 1$ , which is absurd. If  $r_i w \cdot \lambda > w \cdot \lambda$ , then  $r_i r_i w \cdot \lambda < r_i w \cdot \lambda$ , and by the above paragraph,  $l(w) = l(r_i r_i w) > l(r_i w)$ , contrary to hypothesis. Thus,  $r_i w \cdot \lambda < w \cdot \lambda$ . This proves the first statement.

For the second statement, assume that  $\lambda + \rho \in P^+$ , and  $l(r_i w) > l(w)$ . We first show that  $w^{-1} \alpha_i \in \Delta^+$ . Since  $0 \in P^+$ , by the first statement we have  $w \cdot 0 > r_i w \cdot 0$ . Thus, writing  $w \cdot 0 - r_i w \cdot 0 = n \alpha_i$  with  $n > 0$ , we have  $0 - w^{-1} r_i w \cdot 0 = n w^{-1} \alpha_i$ . But  $0 - w^{-1} r_i w \cdot 0 = \langle \Phi_{w^{-1} r_i w} \rangle$ , so that  $n w^{-1} \alpha_i > 0$ . Thus  $w^{-1} \alpha_i \in \Delta^+$ .

Now suppose  $r_i w \cdot \lambda > w \cdot \lambda$ . Then, writing  $r_i w \cdot \lambda - w \cdot \lambda = m \alpha_i$  with  $m > 0$ , we have

$$w^{-1} r_i w \cdot \lambda - \lambda = w^{-1} r_i w(\lambda + \rho) - (\lambda + \rho) = m w^{-1} \alpha_i.$$

But  $w^{-1} r_i w(\lambda + \rho) \in \Pi(\lambda + \rho)$ , so that  $w^{-1} r_i w(\lambda + \rho) \leq \lambda + \rho$ , contradicting the fact that  $w^{-1} \alpha_i \in \Delta^+$ . Therefore  $r_i w \cdot \lambda \leq w \cdot \lambda$ .

We are now ready to give a description of those Verma modules which occur as factors in the terms of the BGG resolution.

**THEOREM 4.10.** *Let  $\mu \in \Pi = \Pi(\Lambda_{\Pi}^-) + \Pi(L(\lambda))$ , where  $\lambda \in P^+$ . Then  $\mu \in S$ , where  $S$  is the equivalence class in  $\Pi$  under  $\sim_{\Pi}$  containing  $\lambda$ , if and only if  $\mu$  is of the form  $w \cdot \nu$ , where  $w \in W$ ,  $\nu \in \Pi$ ,  $\nu + \rho \in P^+$ , and, for some  $n \in \mathbf{Z}_+$ , there exist weights  $\lambda = \nu_0, \nu_1, \dots, \nu_n = \nu$  in  $\Pi$  such that  $\nu_i + \rho \in P^+$  for  $i = 0, \dots, n$ , and  $M(\nu_{i-1})$  and  $M(\nu_i)$  have a common submodule  $M(\chi_i)$  for some  $\chi_i + \rho \in P^+$  for  $i = 1, \dots, n$ . (We do not require the  $\chi_i$ 's to be in  $\Pi$ .)*

**PROOF.** First assume that  $\mu = w \cdot \nu$ , where  $\nu$  is as in the statement of the theorem. By definition of  $\sim_{\Pi}$ ,  $\nu \sim_{\Pi} \lambda$ , so that  $\nu \in S$ . Also, by Lemma 4.9, an easy induction on  $l(w)$  using Lemma 2.7 gives an imbedding  $M(w \cdot \nu) \subseteq M(\nu)$ . Since  $w \cdot \nu \in \Pi$  by Corollary 4.6, we have  $\mu = w \cdot \nu \in S$ .

Conversely, suppose  $\mu \in S$ . Then there are weights  $\lambda = \mu_0, \mu_1, \dots, \mu_n = \mu$  in  $\Pi$  such that  $\mu_0 R_{\Pi} \mu_1 R_{\Pi} \cdots R_{\Pi} \mu_n$ . By induction on  $n$ , we may assume that  $\mu_{n-1}$  is of the form  $w \cdot \nu_{n-1}$ , where  $\nu_{n-1} \in \Pi$  and  $\nu_{n-1} + \rho \in P^+$ , and there exists weights  $\lambda = \nu_0, \nu_1, \dots, \nu_{n-1}$  in  $\Pi$  such that every  $\nu_i + \rho \in P^+$  and every  $M(\nu_i)$  and  $M(\nu_{i-1})$  have a common submodule  $M(\chi_i)$  with  $\chi_i + \rho \in P^+$ . By Lemma 4.8, there is some  $w' \in W$  with  $w'^{-1} \cdot \mu \in \Pi$  and  $w'^{-1} \cdot \mu + \rho \in P^+$ . Set  $\nu_n = w'^{-1} \cdot \mu$ , so that  $\mu = w' \cdot \nu_n$ ,  $\nu_n \in \Pi$ , and  $\nu_n + \rho \in P^+$ . As in the preceding paragraph, there are imbeddings  $M(\mu) \subseteq M(\nu_n)$  and  $M(\mu_{n-1}) \subseteq M(\nu_{n-1})$ . Since  $M(\mu_{n-1})$  and  $M(\mu)$  have a common irreducible subquotient  $L(\chi)$  by definition of  $R_{\Pi}$ , we also have that  $L(\chi)$  is a subquotient of both  $M(\nu_{n-1})$  and  $M(\nu_n)$ . Thus, by Proposition 2.6,  $\text{Hom}(M(\chi), M(\nu_{n-1})) \neq 0$  and  $\text{Hom}(M(\chi), M(\nu_n)) \neq 0$ . We now show, by induction on  $\text{ht}(\lambda - \chi)$ , that for some  $w'' \in W$ ,  $w'' \cdot \chi + \rho \in P^+$  and  $M(w'' \cdot \chi)$  is a submodule of both  $M(\nu_{n-1})$  and  $M(\nu_n)$ , which will finish the proof. This is clear if  $\chi = \lambda$ , or if  $\chi + \rho \in P^+$ . Otherwise, for some  $i = 1, \dots, l$ ,  $r_i \cdot \chi > \chi$ , and hence  $\bar{r}_i \cdot \chi = r_i \cdot \chi$ . Note that  $\bar{r}_i \cdot \nu_{n-1} = \nu_{n-1}$  and  $\bar{r}_i \cdot \nu_n = \nu_n$ , since both  $\nu_{n-1} + \rho$  and

$v_n + \rho$  are in  $P^+$ . Thus, by Proposition 2.11,  $\text{Hom}(M(r_i \cdot \chi), M(v_{n-1})) \neq 0$  and  $\text{Hom}(M(r_i \cdot \chi), M(v_n)) \neq 0$ . Since  $\text{ht}(\lambda - r_i \cdot \chi) < \text{ht}(\lambda - \chi)$ , the induction is complete. Thus, setting  $\chi_n = w'' \cdot \chi$ , we are done.

**5. The character and denominator formulas.** We continue our analysis of the resolution of  $L(\lambda)$ , for  $\lambda \in P^+$ , given in Theorem 4.4. First, we obtain some general results on sums of characters. Next, we obtain an identity involving the multiplicities  $(C_k : M(\mu))$ . Comparison of these results leads to an expression for the character of  $L(\lambda)$  in terms of the characters of Verma modules, which yields analogs of the classical character and denominator formulas of Weyl. We retain the notation of the previous section. It is convenient to introduce a certain directed graph, as follows.

**DEFINITION 5.1.** Let  $G$  be the directed graph whose vertex set is  $\mathfrak{h}^*$ , with an edge of multiplicity  $(M(\nu) : L(\mu))$  from  $\mu$  to  $\nu$  whenever  $\mu < \nu$ . Define  $g_i(\nu, \mu)$  to be the number of directed paths in  $G$  of length  $i$  from  $\mu$  to  $\nu$ , where, by convention, we take  $g_0(\nu, \mu) = \delta(\nu, \mu)$ . We also define  $a(\nu, \mu) = \sum_{i \geq 0} (-1)^i g_i(\nu, \mu)$  for any  $\mu, \nu \in \mathfrak{h}^*$ .

**LEMMA 5.2.** For any  $\mu, \nu \in \mathfrak{h}^*$ ,  $\text{ch } L(\nu) = \sum_{\mu \leq \nu} a(\nu, \mu) \text{ch } M(\mu)$ .

**PROOF.** We first show that for any  $\nu, \chi \in \mathfrak{h}^*$ ,

$$(5.1) \quad \sum_{\chi \leq \mu \leq \nu} a(\nu, \mu) (M(\mu) : L(\chi)) = \delta(\nu, \chi).$$

We have

$$\begin{aligned} \sum_{\chi \leq \mu \leq \nu} a(\nu, \mu) (M(\mu) : L(\chi)) &= \sum_{\chi < \mu \leq \nu} \sum_{i \geq 0} (-1)^i g_i(\nu, \mu) (M(\mu) : L(\chi)) \\ &\quad + \sum_{i \geq 0} (-1)^i g_i(\nu, \chi) (M(\chi) : L(\chi)) \\ &= \sum_{i \geq 0} (-1)^i g_{i+1}(\nu, \chi) + \sum_{i \geq 0} (-1)^i g_i(\nu, \chi) \\ &= g_0(\nu, \chi) = \delta(\nu, \chi), \end{aligned}$$

which proves (5.1). But this yields

$$\begin{aligned} \text{ch } L(\nu) &= \sum_{\chi \leq \nu} \delta(\nu, \chi) \text{ch } L(\chi) \\ &= \sum_{\chi \leq \nu} \left( \sum_{\chi \leq \mu \leq \nu} a(\nu, \mu) (M(\mu) : L(\chi)) \right) \text{ch } L(\chi) \\ &= \sum_{\mu \leq \nu} a(\nu, \mu) \sum_{\chi \leq \mu} (M(\mu) : L(\chi)) \text{ch } L(\chi) \\ &= \sum_{\mu \leq \nu} a(\nu, \mu) \text{ch } M(\mu). \end{aligned}$$

**LEMMA 5.3.** If  $\nu \in \mathfrak{h}^*$  and  $\sum_{\mu \leq \nu} n_\mu \text{ch } M(\mu) = 0$ , where  $n_\mu \in \mathbb{Z}$  for all  $\mu \leq \nu$ , then every  $n_\mu = 0$ .

**PROOF.** Suppose some  $n_\chi \neq 0$ . We may choose this  $\chi$  such that  $\text{ht}(\nu - \chi)$  is minimal. Writing

$$\sum_{\mu \leq \nu} n_\mu \text{ch } M(\mu) = \sum_{\eta \leq \nu} m_\eta e^\eta,$$

by hypothesis all  $m_\eta = 0$ . But then we have

$$0 = m_\chi = \sum_{\mu \leq \nu} n_\mu \dim M(\mu)_\chi = \sum_{\chi \leq \mu \leq \nu} n_\mu \dim M(\mu)_\chi = n_\chi,$$

contradicting the choice of  $\chi$ . Thus all  $n_\mu = 0$ .

REMARK. We may think of the above lemma as a statement of the linear independence of the  $\text{ch } M(\mu)$  over  $\mathbf{Z}$ , even when certain well-defined infinite sums are allowed.

LEMMA 5.4. *Let  $\mu \in S$ , and suppose  $\mu = w \cdot \nu$ , where  $\nu \in S$ ,  $\nu + \rho \in P^+$ , and  $w \in W$ . Then*

$$\sum_{k \geq 0} (-1)^k (C_k : M(\mu)) = (-1)^{l(w)} \sum_{k \geq 0} (-1)^k (C_k : M(\nu)).$$

PROOF. By Theorem 4.4 and Corollary 4.7,

$$\begin{aligned} \sum_{k \geq 0} (-1)^k (C_k : M(\mu)) &= \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\mu \\ &= (-1)^{l(w)} \sum_{k \geq 0} (-1)^k \dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_\nu \\ &= (-1)^{l(w)} \sum_{k \geq 0} (-1)^k (C_k : M(\nu)). \end{aligned}$$

THEOREM 5.5. *Let  $\lambda \in P^+$ . Then*

$$\text{ch } L(\lambda) = \sum_{\nu \in P^+} a(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} \text{ch } M(w \cdot \nu).$$

PROOF. From the resolution

$$\cdots \rightarrow C_k \rightarrow C_{k-1} \rightarrow \cdots \rightarrow C_0 \rightarrow L(\lambda) \rightarrow 0$$

of  $L(\lambda)$  given in Theorem 4.4 an application of the Euler-Poincaré principle gives

$$\begin{aligned} \text{ch } L(\lambda) &= \sum_{k \geq 0} (-1)^k \text{ch } C_k = \sum_{k \geq 0} (-1)^k \sum_{\mu \in S} (C_k : M(\mu)) \text{ch } M(\mu) \\ &= \sum_{\mu \in S} \left( \sum_{k \geq 0} (-1)^k (C_k : M(\mu)) \right) \text{ch } M(\mu). \end{aligned}$$

Now, by Theorem 4.10,  $\mu \in S$  if and only if  $\mu = w \cdot \nu$  for some  $w \in W$  and some  $\nu \in S$  with  $\nu + \rho \in P^+$ . Thus,

$$\begin{aligned} (5.2) \quad \text{ch } L(\lambda) &= \sum_{\substack{\nu \in S \\ \nu + \rho \in P^+}} \sum_{w \in W} \left( \sum_{k \geq 0} (-1)^k (C_k : M(w \cdot \nu)) \right) \text{ch } M(w \cdot \nu) \\ &= \sum_{\substack{\nu \in S \\ \nu + \rho \in P^+}} \sum_{w \in W} (-1)^{l(w)} \left( \sum_{k \geq 0} (-1)^k (C_k : M(\nu)) \right) \text{ch } M(w \cdot \nu), \end{aligned}$$

by Lemma 5.4.

On the other hand, by Lemma 5.2 we have

$$(5.3) \quad \text{ch } L(\lambda) = \sum_{\mu \leq \lambda} a(\lambda, \mu) \text{ch } M(\mu).$$

By Lemma 5.3, since every  $\mu \in \Pi$  satisfies  $\mu \leq \lambda$ , we may equate coefficients in the two expressions (5.2) and (5.3) for  $\text{ch } L(\lambda)$  to obtain

$$\sum_{k \geq 0} (-1)^k (C_k: M(\nu)) = a(\lambda, \nu)$$

for all  $\nu + \rho \in P^+$  with  $\nu \in S$ , and  $a(\lambda, \nu) = 0$  for all  $\nu + \rho \in P^+$  with  $\nu \notin S$ . Thus we may rewrite (5.2) as

$$(5.4) \quad \text{ch } L(\lambda) = \sum_{\nu + \rho \in P^+} \sum_{w \in W} (-1)^{l(w)} a(\lambda, \nu) \text{ch } M(w \cdot \nu).$$

Note that if  $\nu + \rho \in P^+$  but  $\nu \notin P^+$ , then  $\nu(h_i) = -1$  for some  $i = 1, \dots, l$ , and  $r_i \cdot \nu = \nu$ , so that for any  $w \in W$ ,  $\text{ch } M(wr_i \cdot \nu) - \text{ch } M(w \cdot \nu) = 0$ , and hence  $\sum_{w \in W} (-1)^{l(w)} \text{ch } M(w \cdot \nu) = 0$ . Thus, we may delete terms from (5.4) to obtain

$$\begin{aligned} \text{ch } L(\lambda) &= \sum_{\nu \in P^+} \sum_{w \in W} (-1)^{l(w)} a(\lambda, \nu) \text{ch } M(w \cdot \nu) \\ &= \sum_{\nu \in P^+} a(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} \text{ch } M(w \cdot \nu). \end{aligned}$$

From the theorem, we immediately obtain an analog of Kostant's multiplicity formula.

**COROLLARY 5.6.** *If  $\lambda \in P^+$  and  $\mu \leq \lambda$ , then*

$$\dim L(\lambda)_\mu = \sum_{\nu \in P^+} a(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} \mathcal{P}(w(\nu + \rho) - (\mu + \rho)).$$

**COROLLARY 5.7.** *If  $\lambda \in P^+$ , then*

$$\text{ch } L(\lambda) = \frac{\sum_{\nu \in P^+} a(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \nu}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}.$$

**PROOF.** It suffices, by Theorem 5.5, to show that, for any  $\mu \in \mathfrak{h}^*$ ,  $\text{ch } M(\mu) = e^\mu / \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$ . But we have

$$\begin{aligned} \text{ch } M(\mu) &= \sum_{\chi \leq \mu} \dim M(\mu)_\chi e^\chi = \sum_{\chi \leq \mu} \mathcal{P}(\mu - \chi) e^\chi \\ &= \sum_{\beta \in Q^+} \mathcal{P}(\beta) e^{\mu - \beta} = e^\mu \sum_{\beta \in Q^+} \mathcal{P}(\beta) e^{-\beta} \\ &= e^\mu \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots)^{\dim \mathfrak{g}_\alpha} \\ &= e^\mu / \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}. \end{aligned}$$

**COROLLARY 5.8 (DENOMINATOR FORMULA).** *We may express the denominator in Corollary 5.7 as*

$$\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{\nu \in P^+} a(0, \nu) \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \nu}.$$

**PROOF.** Since  $L(0)$  is the trivial module, with  $\text{ch } L(0) = e^0$ , and since  $0 \in P^+$ , we may apply Corollary 5.7 with  $\lambda = 0$  to obtain

$$e^0 = \frac{\sum_{\nu \in P^+} a(0, \nu) \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \nu}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}},$$

from which the result immediately follows.

COROLLARY 5.9 (CHARACTER FORMULA). *If  $\lambda \in P^+$ , then*

$$\text{ch } L(\lambda) = \frac{\sum_{\nu \in P^+} a(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \nu}}{\sum_{\nu \in P^+} a(0, \nu) \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \nu}}.$$

One other consequence of Theorem 5.5 is the following proposition, which gives some partial information about the multiplicities  $(M(\nu) : L(\mu))$ .

PROPOSITION 5.10. *If  $\lambda, \nu \in P^+$ , and  $w \in W$ , then*

$$\sum_{i \geq 0} (-1)^i g_i(\lambda, w \cdot \nu) = (-1)^{l(w)} \sum_{i \geq 0} (-1)^i g_i(\lambda, \nu).$$

*In particular,*

$$\sum_{i \geq 0} (-1)^i g_i(\lambda, w \cdot \lambda) = (-1)^{l(w)}.$$

PROOF. From Lemma 5.2 we have

$$\text{ch } L(\lambda) = \sum_{\mu \leq \lambda} \sum_{i \geq 0} (-1)^i g_i(\lambda, \mu) \text{ch } M(\mu),$$

and from Theorem 5.5,

$$\text{ch } L(\lambda) = \sum_{\nu \in P^+} \sum_{i \geq 0} (-1)^i g_i(\lambda, \nu) \sum_{w \in W} (-1)^{l(w)} \text{ch } M(w \cdot \nu).$$

From the linear independence of the characters of Verma modules given in Lemma 5.3, we may equate coefficients of  $\text{ch } M(w \cdot \nu)$  in these two expressions for  $\text{ch } L(\lambda)$ , giving the first statement. The second statement follows from the first.

**6. A sufficient condition for sharpening the results.** Here we show that, under certain conditions on Verma module imbeddings, we can sharpen both the character and denominator formulas so that they take their classical form. We also show that the BGG resolution takes the same form as in the finite dimensional and symmetrizable cases, and that the radical is zero.

DEFINITION 6.1. We say that  $\mathfrak{g}$  satisfies the Verma imbedding property if, whenever  $\lambda, \mu \in P^+$ ,  $\text{Hom}(M(\mu), M(\lambda)) \neq 0$  implies  $\lambda = \mu$ . We say that  $\mathfrak{g}$  satisfies the strong Verma imbedding property if, whenever  $\lambda + \rho, \mu + \rho \in P^+$ ,  $\text{Hom}(M(\mu), M(\lambda)) \neq 0$  implies  $\lambda = \mu$ .

For example, in [11] it is shown, using [6], that if  $A$  is a symmetrizable GCM, i.e. there is some diagonal matrix  $D$  with positive integers on the diagonal such that  $DA$  is symmetric, then  $\mathfrak{g}$  satisfies the strong Verma imbedding property.

THEOREM 6.2. *Suppose  $\mathfrak{g}$  satisfies the Verma imbedding property. If  $\lambda \in P^+$ , then*

$$\text{ch } L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{l(w)} e^{w \cdot 0}}$$

*and*

$$\Pi_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_{\alpha}} = \sum_{w \in W} (-1)^{l(w)} e^{w \cdot 0}.$$

PROOF. If  $\lambda, \nu \in P^+$  and  $a(\lambda, \nu) \neq 0$ , then  $g_i(\lambda, \nu) \neq 0$  for some  $i \geq 0$ . But then by Proposition 2.6 we would have  $\text{Hom}(M(\nu), M(\lambda)) \neq 0$ , hence  $\lambda = \nu$ . Thus  $a(\lambda, \nu) = 0$  whenever  $\lambda, \nu \in P^+$  and  $\lambda \neq \nu$ . In particular,  $a(0, \nu) = 0$  for all  $\nu \in P^+$  with  $\nu \neq 0$ . Since  $a(\lambda, \lambda) = 1 = a(0, 0)$ , the result follows from Corollaries 5.8 and 5.9.

COROLLARY 6.3. *Let  $\mathfrak{g}$  satisfy the Verma imbedding property. If  $\mathfrak{r}$  is the largest ideal of  $\mathfrak{g}$  such that  $\mathfrak{r} \cap \mathfrak{h} = 0$ , then  $\mathfrak{r} = 0$ .*

PROOF. Let  $\mathfrak{g}' = \mathfrak{g}/\mathfrak{r}$ . Note that  $\mathfrak{g}'$  is also a GCM Lie algebra defined by  $A$ . Thus, Theorem 6.2 applied to  $\mathfrak{g}$  and  $\mathfrak{g}'$  yields

$$\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}'_\alpha}.$$

But this implies that  $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}'_\alpha$  for all  $\alpha \in \Delta^+$ , since otherwise, choosing  $\alpha$  minimal such that  $\dim \mathfrak{g}_\alpha \neq \dim \mathfrak{g}'_\alpha$  and comparing coefficients of  $e^{-\alpha}$ , we would have a contradiction. Therefore  $\mathfrak{r} = 0$ .

THEOREM 6.4. *Let  $\mathfrak{g}$  satisfy the strong Verma imbedding property. If  $\lambda \in P^+$ , then  $L(\lambda)$  has a resolution*

$$\cdots \rightarrow C_k \rightarrow C_{k-1} \rightarrow \cdots \rightarrow C_0 \rightarrow L(\lambda) \rightarrow 0,$$

where each  $C_k$  has a Verma series whose factors are those  $M(w \cdot \lambda)$  with  $l(w) = k$ , each occurring with multiplicity one.

PROOF. We must show that, in the resolution of Theorem 4.4, if  $\mu \in S$  then  $\mu = w \cdot \lambda$  for some  $w \in W$ , and that  $\dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \lambda} = \delta(k, l(w))$ . The first statement follows from Theorem 4.10 and the strong Verma imbedding property. For the second statement, using Lemma 4.5 we have

$$\begin{aligned} \dim(\Lambda \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \lambda} &= \sum_{\substack{\mu \in \Pi(\Lambda \mathfrak{n}^-) \\ \nu \in \Pi(L(\lambda)) \\ \rho + \mu + \nu = w(\lambda + \rho)}} \dim(\Lambda \mathfrak{n}^-)_\mu \dim L(\lambda)_\nu \\ &= \sum_{\substack{\mu \in \Pi(\Lambda \mathfrak{n}^-) \\ \nu \in \Pi(L(\lambda)) \\ \mu + \nu = \lambda}} \dim(\Lambda \mathfrak{n}^-)_\mu \dim L(\lambda)_\nu = \dim(\Lambda \mathfrak{n}^-)_0 \dim L(\lambda)_\lambda = 1. \end{aligned}$$

Thus, there is exactly one value of  $k$  such that  $\dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \lambda} = 1$ , and for all other  $k$ ,  $\dim(\Lambda^k \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \lambda} = 0$ . Since  $\langle \Phi_w \rangle = \rho - w\rho$  and  $|\Phi_w| = l(w)$ , we have  $\dim(\Lambda^{l(w)} \mathfrak{n}^-)_{w\rho - \rho} \geq 1$ . Since  $\dim L(\lambda)_{w\lambda} = 1$  and  $w\rho - \rho + w\lambda = w \cdot \lambda$ , we have  $\dim(\Lambda^{l(w)} \mathfrak{n}^- \otimes_K L(\lambda))_{w \cdot \lambda} \geq 1$ , and the result follows.

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